ELSEVIER

Discrete Mathematics 213 (2000) 29–34

DISCRETE MATHEMATICS

A note on generalized chromatic number and generalized girth

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Received 20 December 1994; revised 10 December 1998; accepted 30 January 1999

Abstract

Erdős proved that there are graphs with arbitrarily large girth and chromatic number. We study the extension of this for generalized chromatic numbers.

MSC Codes: 05C15; 05C65; 05C80

Keywords: Graph coloring; Girth

Generalized graph coloring describes the partitioning of the vertices into classes whose induced subgraphs satisfy particular constraints. When $\boldsymbol{P}$ is a family of graphs, the $\boldsymbol{P}$ *chromatic number* of a graph G, written $\chi_{\boldsymbol{P}}$, is the minimum size of a partition of $V(G)$ into classes that induce subgraphs of G belonging to $\boldsymbol{P}$. When $\boldsymbol{P}$ is the family of independent sets, $\chi_{\boldsymbol{P}}$ is the ordinary chromatic number. General aspects are studied in [1–3,7–9,11–14,17,18]. Many additional results are known about particular generalized chromatic numbers.

One aim in the study of generalized chromatic numbers is the extension of classical coloring results. Erdős [4] proved that there exist graphs of large chromatic number and large girth. We study the extension of this for a class of generalized coloring parameters. We consider the family $\boldsymbol{P}$ consisting of all graphs not containing H as a subgraph; we call the corresponding parameter the H-chromatic number and write it as χ_H.

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[1] Research supported in part by NSA/MSP Grant MDA904-93-H-3040.

PII: S0012-365X(99)00165-X

The natural extension requires an appropriate definition for generalized girth. For $j \geqslant 2$, an *(H,j)-cycle* in a graph G is a list of distinct subgraphs $H_1, \ldots, H_j$, each isomorphic to H, such that $\bigcup_{i=1}^{j} H_i$ contains a cycle that decomposes into j nontrivial paths with the ith path in H_i (any two successive paths in the decomposition share one vertex). The *H-girth* of G, written $g_H(G)$, is the minimum j such that G contains an (H,j)-cycle, if this exists; otherwise $g_H(G) = \infty$.

One might prefer a weaker notion of cycle. For $j \geqslant 2$, a *weak (H,j)-cycle* in G is a list of distinct subgraphs $H_1, \ldots, H_j$, each isomorphic to H, and a selection of distinct vertices $x_1, \ldots, x_j \in V(G)$ such that $x_i \in V(H_{i-1}) \cap V(H_i)$, with subscripts taken modulo j. The *weak* H-girth $g_H^*(G)$ is the minimum j such that G contains a weak (H,j)-cycle, if this exists; otherwise $g_H^*(G) = \infty$.

Every (H,j)-cycle is a weak (H,j)-cycle, so $g_H^*(G) \leqslant g_H(G)$. The weak H-girth may be considerably smaller than the H-girth. Trivial examples arise when H is disconnected. Also, when H is a 3-vertex path and $G = K_{1,3}$, we have $g_H(G) = \infty$ and $g_H^* = 2$.

One naturally seeks the existence of graphs with arbitrarily large H-chromatic number and arbitrarily large H-girth. Such a result does not hold for weak H-girth, even when H is r-edge-connected.

Example. *If H is the union of two copies of K_{r+1} sharing a vertex, then $\chi_H(G) \geqslant 4r+2$ implies $g_H^*(G) \leqslant 2$.* We prove the contrapositive. The union of three $(r+1)$-cliques sharing a single vertex has weak H-girth 2, so $g_H^*(G) > 2$ forbids this as a subgraph. Thus, in G there is no vertex x whose neighborhood induces a subgraph containing three disjoint r-cliques. Thus, in $G[N(x)]$ there is a set S_x of at most $2r$ vertices (the vertices of a maximal set of disjoint r-cliques) that together contain some vertex of each r-clique in $N(x)$.

Let G' be the spanning subgraph of G whose edges include the edges from each x to S_x for each $v \in V(G)$. Since each S_x has size at most $2r$, each m-vertex subgraph of G' has at most $2rm$ edges and thus minimum degree at most $4r$. By the Szekeres–Wilf Theorem, G' has ordinary chromatic number at most $4r + 1$. Every proper ordinary coloring of G' uses at least two colors on every $(r+1)$-clique in G, since $G'[S]$ has no isolated vertex when $G[S]$ is an $(r+1)$-clique. Thus $\chi_{K_{r+1}}(G) \leqslant 4r + 1$. Since H contains K_{r+1}, we have $\chi_H(G) \leqslant \chi_{K_{r+1}}(G) \leqslant 4r + 1$.

Similar examples occur whenever H is not 2-connected, but weak H-girth equals H-girth when H is 2-connected. We prove the desired result using the stronger concept of H-girth. This also follows from the result of Erdős and Hajnal [5] establishing the existence of r-uniform hypergraphs with large girth and chromatic number (constructive proofs appear in [10,16]). If H has order r, then a graph G with H-girth g and H-chromatic number k (where g is sufficiently large in terms of H) can be obtained from an r-uniform hypergraph $\boldsymbol{H}$ with girth g and chromatic number k by taking the union of copies of H defined on the vertex sets of the edges of $\boldsymbol{H}$.

Erdős and Hajnal did not directly define cycles in hypergraphs; instead they said that an r-uniform hypergraph is *s-circuitless* if for $t \leqslant s$ every set of t edges contains at least $1+(r-1)t$ vertices in its union. When H has r vertices, an (H,j)-cycle has at most $(r-1)j$ vertices, so their definition of s-circuitless is equivalent to $g_H(G) > s$ when the edges of the hypergraph correspond to copies of H in G.

Using the strong version of H-girth, we give a short direct existence argument for graphs with large H-chromatic number and large H-girth. When H is a clique, this becomes a proof of the Erdős–Hajnal result. The main idea of the construction is similar to theirs, but the computations are somewhat different, and our presentation is perhaps more self-contained. When H has order r, the order of our graphs with $g_H > s$ and $\chi_H > k$ is about k^{rs}. The minimum order of such graphs is addressed in [6], so the point of our note is its alternative computations.

We summarize the approach. When seeking $g_H(G) > s$, we say that an (H,j)-cycle is a *short H-cycle* if $j \leqslant s$. We will use the 'deletion method', generating an n-vertex graph having many copies of H but few short H-cycles. To do this, we let r-sets receive copies of H with probability p. For appropriate p, we expect so many copies of H that in some graph every set of size $\lceil n/k \rceil$ contains more copies of H than the number of short H-cycles in the graph. After deleting edges in copies of H to break all the short H-cycles, every set of size at least n/k still contains a copy of H. By the pigeonhole principle, the H-chromatic number of the resulting graph exceeds k.

We need a numerical lemma about tail probabilities in the binomial distribution.

Lemma. *If X has the binomial distribution with N trials and success probability p, then* $\mathrm{Prob}(X \leqslant pN/2) < 2(2/e)^{pN/2}$.

Proof. For $1 \leqslant k \leqslant pN/2$, we have

$$\frac{\binom{N}{k}\, p^k(1-p)^{N-k}}{\binom{N}{k-1}\, p^{k-1}(1-p)^{N-k+1}} = \frac{N-k+1}{k}\,\frac{p}{1-p} > 2.$$

In particular, $\mathrm{Prob}(X = pN/2 - k) < 2^{-k}\mathrm{Prob}(X = pN/2)$. Summing, we obtain

$$\mathrm{Prob}(X \leqslant pN/2) < 2\binom{N}{pN/2} p^{pN/2}(1-p)^{(1-p/2)N}.$$

Because $\binom{N}{\alpha N} < (1/\alpha)^{\alpha N}/(1-\alpha)^{(1-\alpha)N}$ and $1-\beta < e^{-\beta}$, we conclude

$$\mathrm{Prob}(X \leqslant pN/2) < 2\cdot 2^{pN/2}\left(\frac{1-p}{1-p/2}\right)^{(1-p/2)N}$$

$$= 2\cdot 2^{pN/2}\left(1 - \frac{p/2}{1-p/2}\right)^{(1-p/2)N} < 2\cdot 2^{pN/2}e^{-pN/2}. \qquad \square$$

Theorem. *Let H be a graph of order r. If s,k are positive integers with $s > r$, then there is a graph G with $g_H(G) > s$ and $\chi_H(G) > k$. Furthermore, if n is sufficiently* *large, then there is a graph G of order n with $g_H(G) > s$ and $\chi_H(G) \geqslant n^{1/(r^\varepsilon s)}$, where $\varepsilon = 1$ if H is 2-connected and $\varepsilon = 2$ if H is not 2-connected.*

Proof. From vertex set $[n]$ we select r-subsets, independently, each with probability p. This yields a random r-uniform hypergraph R.

If H is 2-connected, then set $\ell = s$; otherwise, set $\ell = rs$. A *set-cycle* of length j in G' is a cyclic arrangement of j selected r-sets such that the intersections of successive pairs yield a system of j distinct vertices as representatives. A set-cycle is *short* if it has length at most ℓ. Let X be the number of short H-cycles in R. When n is sufficiently large and N and p are appropriately chosen in terms of n, we claim that $E(X) < pN/4$, and that with probability at least $\frac{1}{2}$ every set of $\lceil n/k \rceil$ vertices contains at least $pN/2$ selected r-sets. Thus, in some R every $\lceil n/k \rceil$-set contains at least $pN/2$ edges (r-sets) of R. Let R' be the hypergraph obtained from R by deleting some edge (r-set) from every short set-cycle in R. On each edge (r-set) in R', place a copy of H, and let G be the graph formed by the union of these copies of H.

If H is 2-connected, then the only copies of H in G are the ones we have placed into the r-sets of R', so $g_H(G) > \ell = s$. If H is not 2-connected, then every cycle in G is either entirely in an r-set in R', or else it goes through all the sets of a set-cycle in R'. In the latter case this cycle has length at least $\ell + 1$, so it is not contained in the union of s subgraphs isomorphic to H. Hence $g_H(G) > s$ in this case as well.

Finally, the H-chromatic number of G is large; since every set of $\lceil n/k \rceil$ vertices of R' contains an edge of R', we have $\chi_H(G) > k$.

All that remains is to justify the inequalities claimed earlier. It suffices to show that our assertions hold when n is sufficiently large and $k = \lfloor n^{1/rs} \rfloor$. Set $N = \binom{n/k}{r}$ and $p = 8(1 + \log k)n/(kN)$. The expected number of $(H,2)$-cycles is less than $(n^2/2)\binom{n}{r-2}^2 p^2$, and the expected number of (H,j)-cycles is less than $(n^j/2j)\binom{n}{r-2}^j p^j$. Letting $\beta = (2/s)n^{rs-s}p^s/(r-2)!^s$, we have

$$E(X) < \frac{n^2}{4}\binom{n}{r-2}p^2 + \sum_{j=2}^{s} \frac{n^j}{2j}\binom{n}{r-2}^j p^j$$

$$\leqslant \frac{n^s}{s}\binom{n}{r-2}^s p^s \leqslant \beta/2.$$

By Markov's inequality, the probability now exceeds $\frac{1}{2}$ that $X \leqslant \beta$.

Consider the $\lceil n/k \rceil$-sets. By the lemma, the probability that *some* $\lceil n/k \rceil$-set of vertices contains at most $pN/2$ selected k-sets is less than α, where $\alpha = 2(2/e)^{pN/2}\binom{n}{\lceil n/k \rceil}$. When we have $k \leqslant n^{1/6}$, we can use Stirling's approximation to conclude that $\alpha < \frac{1}{2}(2/e)^{pN/2}e^{(1+\log k)n/k}$. With $pN = 8(1 + \log k)n/k$, this guarantees $\alpha \leqslant \frac{1}{2}$.

Hence there exists an n-vertex graph G having at most $pN/2$ selected r-sets and having $X \leqslant \beta$. It remains only to prove that $\beta < pN/2 = 4(1 + \log k)n/k$. We need

$p^s \leqslant (2s(1+\log k)/k) n^{-s(r-1)+1}(r-2)!^s$. Using the definition of p, it suffices to have

$$16(\log k)k^{r-1}r!n^{-r+1} \leqslant \left(\frac{2s(1+\log k)}{k}\right)^{1/s}(r-2)!n^{-r+1+1/s}$$

or

$$(16(\log k)k^{r-1}r(r-1))^s \frac{k}{2s(1+\log k)} \leqslant n.$$

This is satisfied by the condition on k in terms of n. □

Let H be a connected graph. We say that G is an *H-forest* if G is a subgraph of a union of copies of H such that any two of the specified copies of H have at most one common vertex and every cycle in G is contained in one of the specified copies of H.

Corollary. *Let H be a graph, and let $\mathscr{F} = \{F_1, \ldots, F_m\}$ be a family such that no F_i is an H-forest. Let $\mathscr{G}$ be the family of graphs containing no graph in $\mathscr{F}$. For every $k \geqslant 1$, there is a graph $G \in \mathscr{G}$ such that $\chi_H(G) = k$.*

Proof. If s is the maximum order of the graphs in $\mathscr{F}$, then every graph constructed for s in the proof of the theorem belongs to $\mathscr{G}$. □

In particular, if m is less than the girth of H, then C_m is not an H-forest, and hence there exist C_m-free graphs with arbitrarily large H-chromatic number. Also, $\chi_{\boldsymbol{P}}$ is unbounded for C_m-free graphs when $\boldsymbol{P}$ is the family of graphs not containing H as an *induced* subgraph.

The Corollary also follows immediately from the result of Nešetřil and Rödl [15] that the class of graphs avoiding a fixed finite set of 2-connected graphs has the vertex Ramsey property. They proved their theorem using the result of Erdős and Hajnal [5].

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